

An infinite family of magnetized Morgan-Morgan relativistic thin disks

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Received: date / Accepted: date

Abstract Applying the Horský-Mitskievitch conjecture to the empty space solutions of Morgan and Morgan due to the gravitational field of a finite disk, we have obtained the corresponding solutions of the Einstein-Maxwell equations. The resulting expressions are simply written in terms of oblate spheroidal coordinates and the solutions represent fields due to magnetized static thin disk of finite extension. Now, although the solutions are not asymptotically flat, the masses of the disks are finite and the energy-momentum tensor agrees with the energy conditions. Furthermore, the magnetic field and the circular velocity show an acceptable physical behavior.

Keywords Exact solutions · Einstein-Maxwell equations · Relativistic disks

1 Introduction

The study of axially symmetric solutions of the Einstein and Einstein-Maxwell field equations corresponding to disklike configurations of matter, apart from its purely mathematical interest, has a clear astrophysical relevance. Indeed, thin disks can be used to model accretion disks, galaxies in thermodynamical equilibrium and the superposition of a black hole and a galaxy. Disk sources for stationary axially symmetric spacetimes with magnetic fields are also of astrophysical importance mainly in the study of neutron stars, white dwarfs and galaxy formation.

Exact solutions that have relativistic static thin disks as their sources were first studied by Bonnor and Sackfield [1] and Morgan and Morgan [2,3]. Subsequently, several classes of exact solutions corresponding to static [4,5,6,7,8,9,10,11,12,13] and

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stationary [14, 15, 16, 17] thin disks have been obtained by different authors, and the superposition of a static or stationary thin disk with a black hole has been considered [18, 19, 20, 21, 22, 23, 24, 25, 26]. Relativistic disks embedded in an expanding FRW universe have been studied in [27], perfect fluid disks with halos in [28], and the stability of thin disks models has been investigated using a first order perturbation of the energy-momentum tensor in [29]. On the other hand, thin disks have been discussed as sources for Kerr-Newman fields [30, 31], magnetostatic axisymmetric fields [32], and conformastatic and conformastationary metrics [33, 34, 35]. Also, models of electrovacuum static counterrotating dust disks were presented in [36], charged perfect fluid disks were studied in [37], and charged perfect fluid disks as sources of static and Taub-NUT-type spacetimes in [38, 39].

Now, the thin disks with magnetic fields presented at references [30, 31, 32] were obtained by means of the well known ‘displace, cut and reflect’ method in order to introduce a discontinuity at the first derivative of one otherwise smooth solution. The result is a solution with a singularity of the delta function type in all the $z = 0$ hypersurface and so can be interpreted as an infinite thin disk. On the other hand, solutions that can be interpreted as thin disks of finite extension can be obtained if a proper coordinate system is introduced. A coordinate system that adapts naturally to a finite source and presents the required discontinuous behavior is given by the oblate spheroidal coordinates. Some examples of finite thin disks from vacuum solutions expressed in these coordinates can be found in references [1, 2, 4, 7], and from electrovacuum solutions in reference [35].

According to the above considerations, the purpose of our paper is to present a new infinite family of exact solutions of the Einstein-Maxwell equations for axially symmetric spacetimes. The solutions are obtained by the use of the Horský-Mitskievitch conjecture [40], which prescribes a quite close connection between isometries of vacuum spacetimes (seed metrics) and the electromagnetic four-potential of a generated Einstein-Maxwell fields. We take the Morgan and Morgan metric disk [2] as the seed solution. The generated solutions describe a family of magnetized finite thin disks, which is the magnetized version of the family of relativistic static Morgan and Morgan disks.

The plan of our paper is as follows. First, in Section 2, the Weyl-Lewis-Papapetrou line element is considered and Einstein-Maxwell equations in cylindrical coordinates are introduced. The procedure to obtain magnetovacuum static, axially symmetric relativistic thin disks without radial pressure is also summarized in this section. Section 3 introduces the Horský-Mitskievitch generating conjecture used to obtain a general kind of Weyl-Lewis-Papapetrou spacetime with magnetic field. In Section 4, we put forward a solution of the Einstein-Maxwell equations describing an infinite family of finite static magnetized thin disks by the use of the Horský-Mitskievitch (HM) conjecture. For this purpose, the well known Morgan and Morgan metric disk [2] is employed as seed solution Φ_s . The analysis of the physical behavior of solutions is presented in Section 5 where the asymptotic behavior of the solutions is examined. Then, we study the behavior of the corresponding energy density, current density, circular velocity and azimuthal pressure. Finally, in Section 6, we conclude with a discussion of our results.

2 Einstein-Maxwell Equations and Thin Disks

The vacuum Einstein-Maxwell equations, in geometrized units such that $c = 8\pi G = \mu_0 = \epsilon_0 = 1$, can be written as

$$G_{ab} = T_{ab}, \quad (1a)$$

$$F^{ab}{}_{;b} = 0, \quad (1b)$$

with the electromagnetic energy-momentum tensor given by

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \quad (2)$$

where

$$F_{ab} = A_{b,a} - A_{a,b}$$

is the electromagnetic field tensor and A_a is the electromagnetic four potential. The line element of a static vacuum spacetime can be written in the Weyl-Lewis-Papapetrou (WLP) form[41]

$$ds^2 = -e^{2\Phi}dt^2 + e^{-2\Phi}[r^2d\varphi^2 + e^{2\Lambda}(dr^2 + dz^2)], \quad (3)$$

where t, φ, r, z are the usual cylindrical coordinates: $-\infty < t, z < \infty$, $0 \leq r$, $0 \leq \varphi \leq 2\pi$, the metric functions Φ and Λ depend only on r and z .

The solutions of the Einstein-Maxwell equations corresponding to a disklike source are even functions of the z coordinate. Therefore, they are continuous functions everywhere but their first z -derivatives discontinuous at the disk surface. Consequently, in order to obtain the energy-momentum tensor and the current density of the source, the jump across the disk of the first z -derivatives of the metric tensor is expressed as

$$b_{ab} = [g_{ab,z}] = 2g_{ab,z}|_{z=0+}, \quad (4)$$

and the jump across the disk of the electromagnetic field tensor is expressed as

$$[F_{za}] = [A_{a,z}] = 2A_{a,z}|_{z=0+}, \quad (5)$$

where the reflection symmetry of the functions with respect to $z = 0$ has been used.

Then, using the distributional approach [42,43,44] or the junction conditions on the extrinsic curvature of thin shells [45,46,47], the Einstein-Maxwell equations yield an energy-momentum tensor as

$$T^{ab} = T_+^{ab}\theta(z) + T_-^{ab}[1 - \theta(z)] + Q^{ab}\delta(z), \quad (6)$$

and the current density as

$$J^a = I^a\delta(z), \quad (7)$$

where $\theta(z)$ and $\delta(z)$ are, respectively, the Heaviside and Dirac distributions with support on $z = 0$. Here T_{\pm}^{ab} are the electromagnetic energy-momentum tensors as they are defined by (2) in the $z \geq 0$ and $z \leq 0$ regions, respectively, whereas

$$Q_b^a = \frac{1}{2}\{b^{az}\delta_b^z - b^{zz}\delta_b^a + g^{az}b_b^z - g^{zz}b_b^a + b_c^c(g^{zz}\delta_b^a - g^{az}\delta_b^z)\}, \quad (8)$$

gives the part of the energy-momentum tensor corresponding to the disk source, and

$$I^a = [F^{az}] \quad (9)$$

is the contribution of the disk source to the current density. The surface energy-momentum tensor, S_{ab} , and the surface current density, j^a , of the disk, can be obtained through the relations

$$S_{ab} = \int Q_{ab} \delta(z) ds_n = e^{\Lambda-\Phi} Q_{ab}, \quad (10)$$

and

$$j_a = \int I_a \delta(z) ds_n = e^{\Lambda-\Phi} I_a, \quad (11)$$

where $ds_n = \sqrt{g_{zz}} dz$ is the physical measurement of length in the normal direction of the disk.

Now, we choose the magnetic potential as

$$A_a = (0, A(r, z), 0, 0), \quad (12)$$

so that we have a pure axially symmetric magnetic field

$$F_{\varphi z} = -A_{,z}, \quad F_{r\varphi} = A_{,r}. \quad (13)$$

Thus, for the metric (3), the only non-zero components of S_b^a and j_a are

$$\begin{aligned} S_0^0 &= 2e^{\Phi-\Lambda} \{A_{,z} - 2\Phi_{,z}\}, \\ S_1^1 &= 2e^{\Phi-\Lambda} A_{,z}, \\ j_\varphi &= -2e^{\Phi-\Lambda} A_{,z}, \end{aligned} \quad (14)$$

where all the quantities are evaluated at $z = 0^+$.

With the orthonormal tetrad

$$e_{(b)}^a = \{V^a, W^a, X^a, Y^a\}, \quad (15)$$

where

$$V^a = e^{-\Phi}(1, 0, 0, 0), \quad (16a)$$

$$W^a = \frac{e^\Phi}{r}(0, 1, 0, 0), \quad (16b)$$

$$X^a = e^{\Phi-\Lambda}(0, 0, 1, 0), \quad (16c)$$

$$Y^a = e^{\Phi-\Lambda}(0, 0, 0, 1), \quad (16d)$$

we can write the surface energy-momentum tensor of the disk in the canonical form as

$$S^{ab} = \epsilon V^a V^b + p W^a W^b. \quad (17)$$

Here ϵ and p are the energy density and the azimuthal pressure of the disk, respectively. We also consider the mass density on the surface of the disk, defined as

$$\mu = \epsilon + p. \quad (18)$$

In the same way, the current density on the disk can be written as

$$j^a = j W^a. \quad (19)$$

As we can see the charge density on the disk surface is identically equal to zero. This can be checked by using Eqs. (9), (11), (12) and (19). The electromagnetic field is of a magnetic type, which can be demonstrated by the electromagnetic invariant

$$F_{ab}F^{ab} = \frac{2e^{(2\Phi-A)}}{r^2}[A_{,r}^2 + A_{,z}^2] \geq 0. \quad (20)$$

We consider now, on the basis of Refs. [12] and [39], the possibility that the energy-momentum tensor S^{ab} and the current density j^a can be interpreted as the superposition of two counterrotating fluids. In order to do this, we cast

$$S^{ab} = \epsilon_+ U_+^a U_+^b + \epsilon_- U_-^a U_-^b, \quad (21a)$$

$$j^a = \sigma_+ U_+^a + \sigma_- U_-^a, \quad (21b)$$

where

$$\epsilon_+ = \epsilon_- = (\epsilon - p)/2, \quad (22a)$$

$$\sigma_+ = -\sigma_- = \frac{j e^\Phi}{2r} \sqrt{\frac{\epsilon}{p} - 1}, \quad (22b)$$

are the energy densities and charge densities of the two counterrotating fluids. The counterrotating velocity vectors are given by

$$U_\pm^a = \frac{V^a \pm U W^a}{\sqrt{1 - U^2}}, \quad (23)$$

where

$$U^2 = \frac{p}{\epsilon} \leq 1 \quad (24)$$

is the counterrotating tangential velocity. Therefore, we have two counterrotating charged fluids with equal energy densities and equal but opposite charge densities.

3 The Horský-Mitskievitch conjecture and the WLP metric

3.1 The seed WLP metric

The WLP line element of a static vacuum spacetime can be written as [41]

$$ds^2 = -e^{2\Phi_s} dt^2 + e^{-2\Phi_s} [r^2 d\varphi^2 + e^{2\Lambda_s} (dr^2 + dz^2)], \quad (25)$$

where $\{t, \varphi, r, z\}$ are the usual cylindrical coordinates: $-\infty < t, z < \infty$, $0 \leq r$, $0 \leq \varphi \leq 2\pi$. The metric functions Φ_s and Λ_s depend on r and z . This metric admits two Killing vectors:

$$\xi_t = (1, 0, 0, 0), \quad (26a)$$

$$\xi_\varphi = (0, 1, 0, 0), \quad (26b)$$

which, in the vectorial space ∂_i are,

$$\xi_t = \delta_t^j \partial_j, \quad (27a)$$

$$\xi_\varphi = \delta_\varphi^j \partial_j, \quad (27b)$$

or, in terms of their components are

$$\xi_t^j = \delta_t^j, \quad (28a)$$

$$\xi_\varphi^j = \delta_\varphi^j, \quad (28b)$$

where δ_b^a is the usual Kronecker delta tensor.

In terms of the one-forms $d\mathbf{t}$ and $d\boldsymbol{\varphi}$ we have

$$\xi_t = -e^{2\Phi_s} d\mathbf{t}, \quad (29a)$$

$$\xi_\varphi = r^2 e^{-2\Phi_s} d\boldsymbol{\varphi}, \quad (29b)$$

where all the orthonormal bases employed are always chosen as a generalization of the set

$$\omega^{(0)} = e^{\Phi_s} d\mathbf{t}, \quad (30a)$$

$$\omega^{(1)} = r e^{-\Phi_s} d\boldsymbol{\varphi}, \quad (30b)$$

$$\omega^{(2)} = r e^{\Lambda_s - \Phi_s} d\mathbf{r}, \quad (30c)$$

$$\omega^{(3)} = r e^{\Lambda_s - \Phi_s} d\mathbf{z}, \quad (30d)$$

the simplest tetrad that can be used for the Weyl solution.

3.2 The WLP solution with a magnetic field

The present section is devoted to the derivation of solutions of the Einstein-Maxwell equations via HM conjecture. Following the method used in [48], we modify the line element (25) to the form

$$ds^2 = -f(r, z)^2 e^{2\Phi_s} dt^2 + \frac{e^{-2\Phi_s} r^2}{f(r, z)^2} d\varphi^2 + f(r, z)^2 e^{2(\Lambda_s - \Phi_s)} (dr^2 + dz^2), \quad (31)$$

that admits two Killing vectors

$$\xi_t = -f(r, z)^2 e^{2\Phi_s} d\mathbf{t}, \quad (32a)$$

$$\xi_\varphi = \frac{r^2 e^{-2\Phi_s}}{f(r, z)^2} d\boldsymbol{\varphi}, \quad (32b)$$

with $f(r, z)$ an arbitrary function.

The Killing vectors ξ and the electromagnetic four-potential \mathbf{A} satisfy

$$\star d\star d\xi = 0, \quad \star d\star d\mathbf{A} = 0, \quad (33)$$

where the \star is the usual Hodge (star) operation. Then, the Killing vector ξ_φ induces the four-potential

$$\mathbf{A} = qf(r, z)\xi_\varphi, \quad (34)$$

or,

$$\mathbf{A} = \frac{qr^2}{fe^{2\Phi_s}} d\varphi. \quad (35)$$

It can be verified through standard calculations that the sourceless Einstein-Maxwell equations are fulfilled if

$$f(r, z) = 1 + c_1 f_1(r, z), \quad (36)$$

where the function f_1 must be a solution of the differential equation

$$G_{(a)}^{(a)} = -R = 0, \quad (37)$$

given the fact, for electrovacuum spacetimes, the Einstein tensor, $G_{(a)(b)}$, is traceless. As in [48] we can choose $c_1 = q^2$ and $f_1 = r^2 e^{-2\Phi_s}$, and then we have the four-potential

$$\mathbf{A} = \frac{qr^2}{q^2 r^2 + e^{2\Phi_s}} d\varphi, \quad (38)$$

the q parameter characterizing the strenght of the electromagnetic field. Under the conditions described above the metric set by (31) is reduced to

$$ds^2 = -e^{2\Phi} dt^2 + e^{-2\Phi} [r^2 d\varphi^2 + e^{2\Lambda} (dr^2 + dz^2)], \quad (39)$$

where, $fe^{\Phi_s} = e^{\Phi}$ and $f^2 e^{\Lambda_s} = e^{\Lambda}$.

Using (14), (15) and (17) we find out the surface energy density

$$\epsilon = \frac{4E^2(r)e^{3\Phi_s - \Lambda_s}(1 - r\Phi_{s,r})\Phi_{s,z}}{(e^{2\Phi} + q^2 r^2)^3}, \quad (40)$$

the azimuthal pressure

$$p = \frac{4rE(r)e^{3\Phi_s - \Lambda_s}[2q^2 r + E(r)\Phi_{s,r}]\Phi_{s,z}}{(e^{2\Phi} + q^2 r^2)^3}, \quad (41)$$

the surface mass density

$$\mu = \frac{4E(r)e^{3\Phi_s - \Lambda_s}\Phi_{s,z}}{(e^{2\Phi} + q^2 r^2)^2}, \quad (42)$$

the surface current density

$$j = \frac{4qre^{4\Phi_s - \Lambda_s}\Phi_{s,z}}{(e^{2\Phi} + q^2 r^2)^2}, \quad (43)$$

and the circular velocity

$$U^2 = U_s^2 \left[1 + \frac{2q^2 r}{E(r)\Phi_{s,r}} \right], \quad (44)$$

where $E(r) = e^{2\Phi_s} - q^2 r^2$, and

$$U_s^2 = \frac{r\Phi_{s,r}}{1 - r\Phi_{s,r}} \quad (45)$$

is the circular velocity of the unmagnetized source, i.e. $q = 0$.

As we can see from (29a), (29b), (32a) and (32b), the method is a solution generating technique that changes the norm of the two Killing vectors without adding twist. Besides, the possible connection of the HM conjecture with inner symmetries of Einstein-Maxwell equations would connect the conjecture with a set of generating methods elaborated by Ernst and other authors for axisymmetric fields [49].

4 Magnetized Morgan-Morgan disks

Let's restrict the previous general model to obtain a solution of the Einstein-Maxwell equations describing an infinite family of finite static magnetized thin disks. For this purpose, we use, as a seed solution (Φ_s, Λ_s) , the well known Morgan and Morgan metric disk (ϕ, λ) [2],

$$\phi(x, y) = - \sum_{n=0}^{\infty} C_{2n} q_{2n}(x) P_{2n}(y),$$

where C_{2n} are arbitrary constants that must be properly specified so that a particular solution may be set. $P_n(y)$ are the usual Legendre polynomials and $q_n(x) = i^{n+1} Q_n(ix)$. $Q_n(z)$ being the Legendre functions of second kind (see [50] and, for the Legendre functions of imaginary argument, [51] pag. 1328). The x and y are the oblate spheroidal coordinates related with the cylindrical coordinates by the relations [51]

$$r^2 = a^2(1 + x^2)(1 - y^2), \quad (46a)$$

$$z = axy, \quad (46b)$$

where $0 \leq x < \infty$ and $-1 \leq y < 1$. The disk has coordinates $x = 0$, $0 \leq y^2 < 1$ and, when the disk is crossed, the sign of y changes, but not its absolute value.

We use the constants C_{2n} as they have been determined by González and Reina in [52]

$$C_{2n} = \frac{mG}{2a} \left[\frac{\pi^{1/2}(4n+1)(2l+1)!}{2^{2l}(2n+1)(l-n)!\Gamma(l+n+\frac{3}{2})q_{2n+1}(0)} \right],$$

for $n \leq l$ and $C_{2n} = 0$ for $n > l$. Consequently, these solutions correspond to the magnetized version of the well known Morgan-Morgan disk [2]. For instance, the first three members of the family of the seed metric functions are given by

$$\phi_1(x, y) = -\frac{mG}{a} [\cot^{-1} x + A(3y^2 - 1)], \quad (47a)$$

$$\phi_2(x, y) = -\frac{mG}{a} [\cot^{-1} x + \frac{10A}{7}(3y^2 - 1) + B(35y^4 - 30y^2 + 3)], \quad (47b)$$

$$\phi_3(x, y) = -\frac{mG}{a} [\cot^{-1} x + \frac{10A}{6}(3y^2 - 1) + \frac{21B}{11}(35y^4 - 30y^2 + 3) + C(231y^6 - 315y^4 + 105y^2 - 5)], \quad (47c)$$

where

$$A = \frac{1}{4}[(3x^2 + 1) \cot^{-1} x - 3x],$$

$$B = \frac{3}{448}[(35x^4 + 30x^2 + 3) \cot^{-1} x - 35x^3 - \frac{55}{3}x],$$

$$C = \frac{5}{8448}[(231x^6 + 315x^4 + 105x^2 + 5) \cot^{-1} x - 231x^5 - 238x^3 - \frac{231}{5}x],$$

which similar, but more involved, expressions for greater values of l .

On the other hand, according to (40), the energy density of the magnetized disk can be written as

$$\epsilon_n = \frac{4E^2 e^{3\phi_n - \lambda_n} [y + (1 - y^2)\phi_{n,y}]\phi_{n,x}}{ay^2[e^{2\phi_n} + q^2 a^2(1 - y^2)]^3}, \quad (49)$$

with

$$E(y) = e^{2\phi_n} - q^2 a^2 (1 - y^2),$$

while, according to (41) and (42) the pressure, p , and mass density, μ , in the surface of the disk are

$$p_n = \frac{4E(1 - y^2)e^{3\phi_n - \lambda_n} \phi_{n,x} [2q^2 a^2 y - \phi_{n,y} E]}{ay^2 [e^{2\phi_n} + q^2 a^2 (1 - y^2)]^3}, \quad (50)$$

and

$$\mu_n = \frac{4Ee^{3\phi_n - \lambda_n} \phi_{n,x}}{ay[e^{2\phi_n} + q^2 a^2 (1 - y^2)]^2}, \quad (51)$$

respectively, and, according to (43) and (44), we have the following surface current density and circular velocity

$$j = \frac{4q(1 - y^2)^{1/2} e^{4\phi_n - \lambda_n} \phi_{n,x}}{y[e^{2\phi_n} + q^2 a^2 (1 - y^2)]^2}, \quad (52)$$

and

$$U_n^2 = U_s^2 \left[1 - \frac{2q^2 a^2 y}{E \phi_{n,y}} \right], \quad (53)$$

respectively, where

$$U_s^2 = -\frac{(1 - y^2) \phi_{n,y}}{y + (1 - y^2) \phi_{n,y}},$$

is the circular velocity of the unmagnetized source, i.e. $q = 0$. The non zero components of the magnetic field on the surface of the disk, $B_z = -A_{\varphi,r}$ and $B_r = A_{\varphi,z}$, are

$$B_z = -\frac{2qa(1 - y^2)^{1/2} e^{2\phi_n} [y + (1 - y^2) \phi_{n,y}]}{y[e^{2\phi_n} + q^2 a^2 (1 - y^2)]^2}, \quad (54)$$

and

$$B_r = -\frac{2qa(1 - y^2) e^{2\phi_n} \phi_{n,x}}{y[e^{2\phi_n} + q^2 a^2 (1 - y^2)]^2}, \quad (55)$$

where we have used the magnetic potential of (35), and all the quantities are evaluated on the disk.

5 The behavior of the solutions

In order to elucidate the behavior of the different particular models, firstly we introduce the dimensionless energy surface density on the disks, defined for $0 \leq \tilde{r} \leq 1$ as $\epsilon_n(r) = a\tilde{\epsilon}_n(\tilde{r})$, where the dimensionless radial variable $r = a\tilde{r}$ has been introduced. In figure 1 the dimensionless surface energy densities $\tilde{\epsilon}_n$ for the models corresponding to $n = 1, 2$ and 3 are depicted. In each case, $\tilde{\epsilon}_n(\tilde{r})$ for $0 \leq \tilde{r} \leq 1$ with $m = 0.10$ for different values of the parameter $\tilde{q} = aq$ are shown. First, we set $\tilde{q} = 0.30$, the bottom curve in each graphic, and then $0.20, 0.10$ and $\tilde{q} = 0$ (dotted curve), the vacuum case. In all the cases, it can be seen that the energy density is everywhere positive and that it vanishes at the edge of the disk. The disks with higher values of n and \tilde{q} show an energy distribution that is more concentrated in the center and lower at the edge. It can be observed that the energy density in the central region of the disk decreases with the presence of a magnetic field.

The figure 2 shows the dimensionless azimuthal pressure, $p_n(r) = a\tilde{p}_n(\tilde{r})$, for the models corresponding to $n = 1, 2$ and 3 . As we see, in all the cases, the pressure is everywhere positive and rapidly increases as we move away from the center of the disk. It reaches a maximum and, later, it rapidly decreases. It can be observed that the presence of a magnetic field increases the pressure everywhere on the disk. Otherwise, the maximum increases and moves away from the disk edge as n increases. It can be also observed that the surface current density, represented in figure 4, has a behavior similar to that of the pressure.

In order to illustrate graphically the behavior of the circular velocities or rotation curves, we introduce the dimensionless quantity $U_n(r) = a\tilde{U}_n(\tilde{r})$. In figure 3 we plot the dimensionless rotation curves for the models corresponding to $n = 1, 2$ and 3 . The circular velocity corresponding to $n = 1$ is a monotonously increasing function of the radius. On the other hand, for $n = 2$ and $n = 3$, the circular velocity increases from a value of zero at the center of the disks until it attains a maximum at a critical radius and then, it decreases to acquire a finite value at the edge of the disk. It can also be seen that the value of the critical radius increases as the values of n decreases and \tilde{q} increases. From which, we can conclude that the magnetic field decreases the circular velocity everywhere on the disk. We study these solutions with other values of the parameters as well, but, in all the cases, we find a similar behavior. We see that the magnitude of m and \tilde{q} are limited uniquely by the condition that the magnitude of the velocities of the disk cannot exceed the speed of light. The explicit calculation of the Kretschmann invariant $R_{abcd}R^{abcd}$ is too lengthy to be reproduced here. A direct computation by the use of REDUCE shows that the first member of the family of the magnetized Morgan-Morgan disk, $n = 1$, has a singularity at the rim of the disk where the Kretschmann invariant becomes infinite although its mass density is finite everywhere, when all the $n > 1$ disks are regular. Obviously, this property in the curvature is, of course, inherited from the seed Morgan and Morgan metric disk (see [53]).

The solution of (39) in terms of the seed solution (Morgan and Morgan metric disk, $\Phi_s = \phi$) can be written in the form

$$\Phi = \ln[e^{2\Phi_s} + q^2 r^2] - \Phi_s. \quad (56)$$

Therefore, when we take $r \rightarrow \infty$ clearly we see that, unlike the Morgan and Morgan seed solution, generated spacetimes are not asymptotically flat. Consequently, although

the technique described here produces a new class of solution of Einstein-Maxwell's equations and all the physical quantities exhibit an excellent behavior, the method offers very little insight into the physical implications of the metrics. The reason of this non-flatness far away from the disk should be that the magnetic field itself does not decay at infinity as it appears to be easily expressible from (20). On the other hand, this still appears to be a viable solution of the Einstein-Maxwell system but, apparently, corresponds to magnetized disks immersed in a Melvin magnetic universe.

6 Concluding remarks

We have presented an infinite family of new exact solutions of the vacuum Einstein-Maxwell equations for static and axially symmetric spacetimes. The solutions describe an infinite family of magnetized finite thin disks, the magnetized version of the family of Morgan and Morgan relativistic thin disks [2]. The first member of the family of the magnetized Morgan-Morgan disk, $n = 1$, has a singularity at the rim of the disk where the Kretschmann invariant becomes infinite, although its mass density is finite everywhere. Whereas all the $n > 1$ disks are regular. Obviously, this property in the curvature is, of course, inherited from the seed Morgan and Morgan metric disk.

Unlike the Morgan and Morgan seed solution, the generated spacetimes are not asymptotically flat. Consequently, although the technique described here produces a new class of solution of Einstein-Maxwell's equations and all the physical quantities exhibit an excellent behavior and the energy-momentum tensor is in fully agreement with all the energy conditions, the method offers very little insight into the physical implications of the metrics. However, these are the first fully integrated exact solutions for this kind of magnetized thin disk sources. They are important in so far as they represent a new family of exact solutions of the Einstein-Maxwell vacuum equations. Moreover, the outlined method may serve as a guideline to find more physically acceptable solutions in future works.

Appendix

We pass on to the definition of the Hodge \star (star) operation. Let's us M a m -dimensional manifold endowed with a metric g . The Hodge \star (star) operation is a map: $\Omega^r(M) \rightarrow \Omega^{r-m}(M)$ whose action, on the basis of the vector $\Omega^r(M)$, is defined by [54]

$$\star(dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_r}) = \frac{\sqrt{|g|}}{(m-r)!} \varepsilon^{\mu_1 \mu_2 \dots \mu_r}_{\mu_{r+1} \dots \mu_m} dx^{\mu_{r+1}} \wedge \dots \wedge dx^{\mu_m}.$$

For a r -form

$$A = \frac{1}{r!} A_{\mu_1 \mu_2 \dots \mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_r} \in \Omega^r$$

we have

$$\star A = \frac{\sqrt{|g|}}{r!(r-m)!} A_{\mu_1 \mu_2 \dots \mu_r} \varepsilon^{\mu_1 \mu_2 \dots \mu_r}_{\mu_{r+1} \dots \mu_m} dx^{\mu_{r+1}} \wedge \dots \wedge dx^{\mu_m} \in \Omega^{m-r},$$

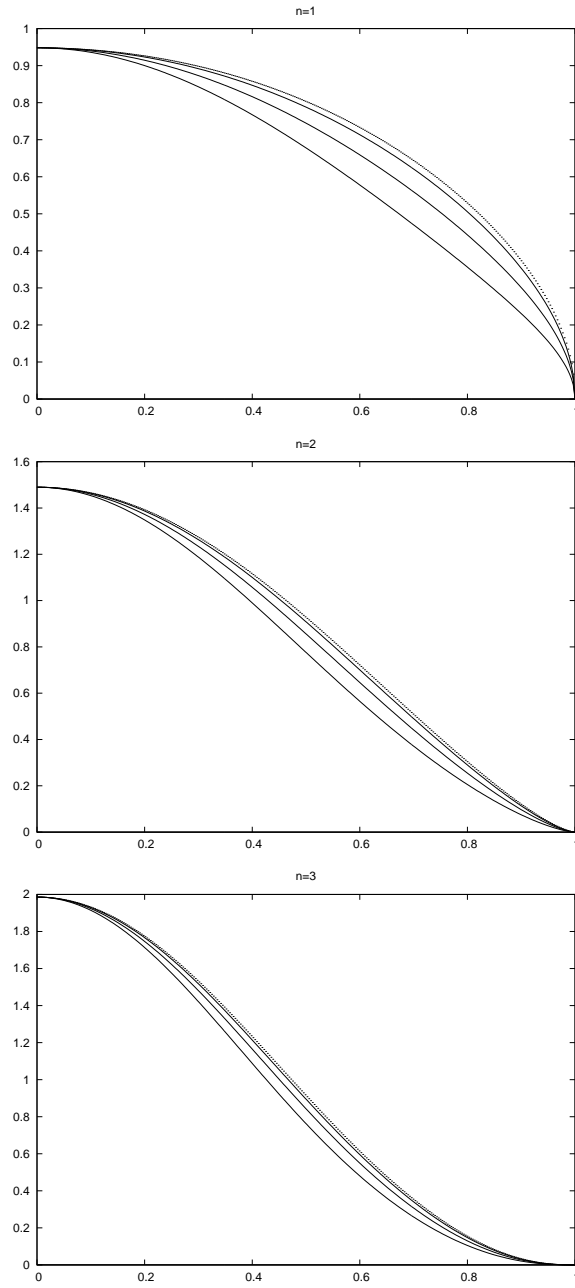


Fig. 1 Dimensionless energy density $\tilde{\epsilon}_n$ as a function of \tilde{r} for the first three disk models with $n = 1, 2, 3$. In each case, we plot $\tilde{\epsilon}_n(\tilde{r})$ for $0 \leq \tilde{r} \leq 1$ with $m = 0.10$ for different values of the parameter \tilde{q} . First, we take $\tilde{q} = 0.30$, the bottom curve in each plot, and then 0.20, 0.10 and $\tilde{q} = 0$ (curve with dots), the vacuum case.

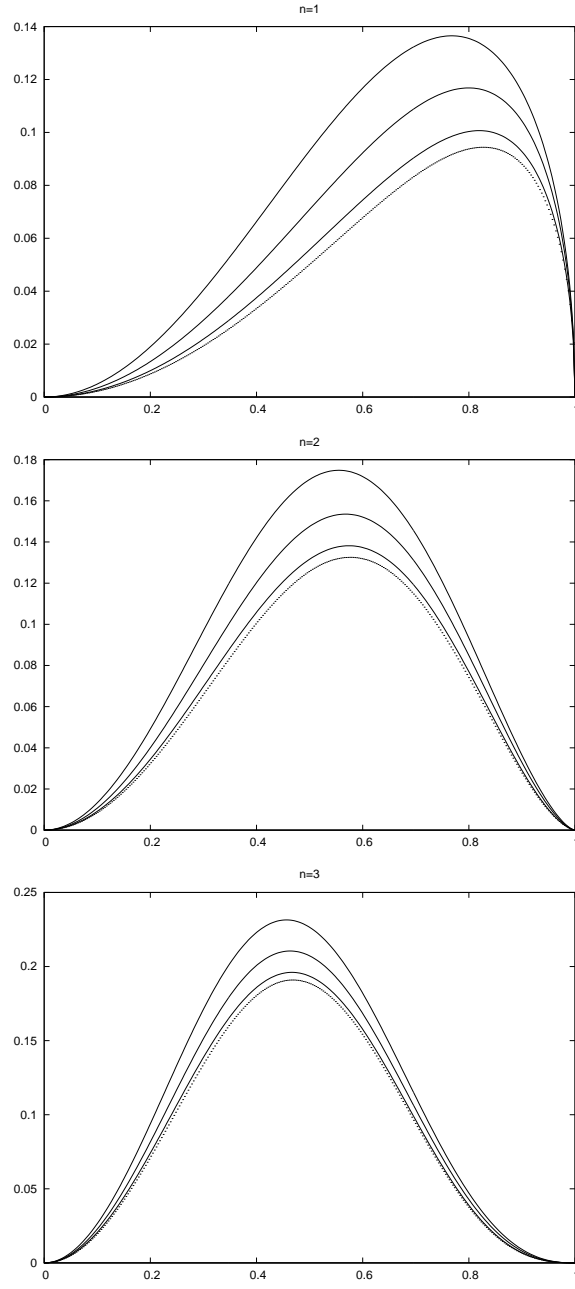


Fig. 2 Dimensionless azimuthal pressure \tilde{p}_n as a function of \tilde{r} for the first three disk models with $n = 1, 2, 3$. In each case, we plot $\tilde{p}_n(\tilde{r})$ for $0 \leq \tilde{r} \leq 1$ with $m = 0.10$ for different values of the parameter \tilde{q} . First, we take $\tilde{q} = 0.30$, the top curve in each plot, and then 0.20, 0.10 and $\tilde{q} = 0$ (curve with dots), the vacuum case.

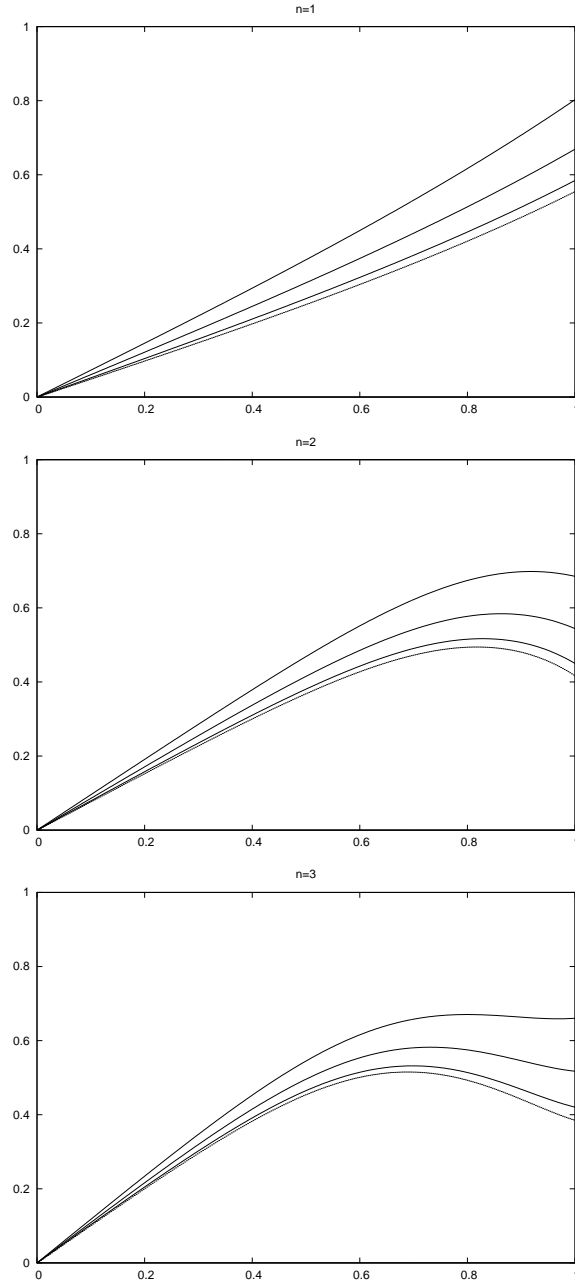


Fig. 3 Dimensionless circular velocity \tilde{U}_n as a function of \tilde{r} for the three first disk models with $n = 1, 2, 3$. In each case, we plot $\tilde{U}_n(\tilde{r})$ for $0 \leq \tilde{r} \leq 1$ with $m = 0.10$ for different values of the parameter \tilde{q} . We first take $\tilde{q} = 0.30$, the top curve in each plot, and then 0.20, 0.10 and $\tilde{q} = 0$ (curve with dots), the vacuum case.

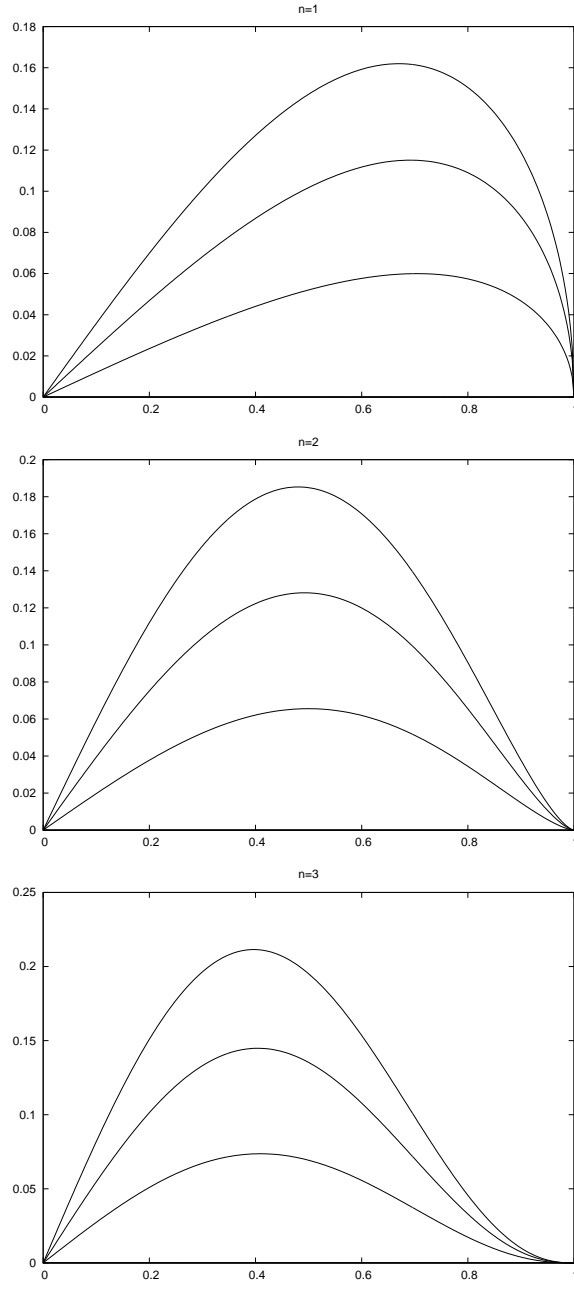


Fig. 4 Dimensionless surface current density \tilde{j}_n as a function of \tilde{r} for the first three disk models with $n = 1, 2, 3$. In each case, we plot $\tilde{j}_n(\tilde{r})$ for $0 \leq \tilde{r} \leq 1$ with $m = 0.10$ for different values of the parameter \tilde{q} . First, we take $\tilde{q} = 0.30$, the top curve in each plot, and then 0.20 and 0.10

where the totally anti-symmetric tensor ε is

$$\varepsilon_{\mu_1\mu_2\ldots\mu_m} = \begin{cases} +1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \\ & \text{is an even permutation of } (12\ldots m), \\ -1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \\ & \text{is an odd permutation of } (12\ldots m), \\ 0 & \text{otherwise,} \end{cases}$$

and “ \wedge ” is the usual exterior product or wedge product.

Acknowledgements A. C. G-P. wants to acknowledge financial support from COLCIEN-CIAS, Colombia.

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